Exam 1 Practice

February 27, 2015

Exercise 1

For any two events A and B,

$$P(A|B) = P(A)$$

.

(a) TRUE

(b) FALSE

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For any two events A and B,

$$P(A|B) = P(A)$$

.

(a) TRUE

(b) FALSE

$P(A|B) = P(A)$ only if A and B are independent. We saw many examples where $P(A|B) \neq P(A)$.

Exercise 2

For two independent events A and B,

$$P(A \cap B) = P(A)P(B)$$

.

(a) TRUE

(b) FALSE

Exercise 2

For two independent events A and B,

$$P(A \cap B) = P(A)P(B)$$

.

(a) TRUE

(b) FALSE

If A and B are independent, which by definition means $P(A|B) = P(A)$, then we have $P(A \cap B) = P(A)P(B)$.

This is how we verify if two events are independent: by checking if $P(A \cap B) = P(A)P(B)$!

Exercise 3

1. Suppose a family has **two** children.
   A= event that the family has at least one boy and at least one girl.
   B= the event that the family has at most one girl.

   Are A and B independent?

2. Suppose a family has **three** children.

   Are A and B (defined exactly as above) independent?

Exercise 3

(a) (1) independent, (2) not independent
(b) (1) not independent, (2) independent
(c) (1) and (2) both independent
(d) (1) and (2) both not independent
(e) I don't know

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(e) I don't know

Exercise 3, part (1) Solution

The sample space is $\Omega = \{BB, GG, BG, GB\}$.

Event A (at least one boy and at least one girl) is

$$A = \{BG, GB\}, \text{ thus } P(A) = \frac{|A|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}.$$

Event B (at most one girl) is

$$B = \{BB, BG, GB\}, \text{ thus } P(B) = \frac{|B|}{|\Omega|} = \frac{3}{4}.$$

Now

$$A \cap B = A, \text{ thus } P(A \cap B) = \frac{1}{2}.$$

So

$$P(A \cap B) \neq P(A)P(B).$$

So A and B are NOT independent.

Exercise 3, part (2) Solution

The sample space is
$\Omega = \{BBB, GGG, BGG, GGB, BBG, GBB, GBG, BGB\}$.

Event A (at least one boy and at least one girl) is

$$A = \{BGG, GGB, BBG, GBB, GBG, BGB\}, \text{ thus } P(A) = \frac{|A|}{|\Omega|} = \frac{6}{8} = \frac{3}{4}.$$

Event B (at most one girl) is

$$B = \{BBB, BBG, GBB, BGB\}, \text{ thus } P(B) = \frac{|B|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}.$$

So $P(A)P(B) = \frac{3}{8}$. Now

$$A \cap B = \{BBG, GBB, BGB\} \text{ thus } P(A \cap B) = \frac{3}{8}.$$

So

$$P(A \cap B) = P(A)P(B).$$

So A and B ARE independent.

Exercise 4

Compute the improper integral

$$\int_0^\infty \frac{1}{x^5}\,dx$$

.

Exercise 4

$$\int_0^\infty \frac{1}{x^5}\,dx$$

(a) $= \frac{1}{4}$
(b) $= -\frac{1}{4}$
(c) $= 0$
(d) diverges
(e) I don't know.

Exercise 4

$$\int_0^\infty \frac{1}{x^5}\,dx$$

(a) $= \frac{1}{4}$
(b) $= -\frac{1}{4}$
(c) $= 0$
(d) diverges
(e) I don't know.

Exercise 4 solution

The integral

$$\int_0^\infty \frac{1}{x^5}\,dx$$

is improper for 2 reasons: it has an infinite discontinuity at 0 and the upper limit of integration is unbounded. Thus, we need to split it up.

$$\int_0^\infty \frac{1}{x^5}\,dx = \int_0^1 \frac{1}{x^5}\,dx + \int_1^\infty \frac{1}{x^5}\,dx$$

Looking at the first summand,

$$\int_0^1 \frac{1}{x^5}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{x^5}\,dx = \lim_{b\to 0^+}\left(-\frac{1}{4}+\frac{1}{4b^4}\right) = \infty.$$

Since the first summand diverges, the whole integral diverges. (Note that the second summand does converge but that doesn't help.)

Exercise 5

Write down the form of the partial fraction decomposition of the following rational function. **Do not** solve for the unknown variables, just stop there.

$$\frac{x^4+3}{x(x^2+2)^2(6x^2-x)}.$$

Exercise 5

I got .... terms in my decomposition

(a) 3

(b) 4

(c) 5

(d) 6

(e) 7

Exercise 5

I got .... terms in my decomposition

(a) 3
(b) 4
(c) 5
(d) 6
(e) 7

Exercise 5 Solution

First, we factor the denominator of

$$\frac{x^4 + 3}{x(x^2 + 2)^2(6x^2 - x)}$$

into linear terms and irreducible quadratics:

$$x(x^2 + 2)^2(6x^2 - x) = x^2(x^2 + 2)^2(6x - 1),$$

where now all the terms are irreducible. Thus the correct form of the partial fraction decomposition is

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 2} + \frac{Ex + F}{(x^2 + 2)^2} + \frac{G}{6x - 1}.$$

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Two cards are chosen at random from a pack of 52 playing cards. What is the probability that at least one of them is a Heart?

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(a) $\frac{22}{51}$

(b) $\frac{15}{34}$

(c) $\frac{19}{34}$

(d) $\frac{29}{51}$

(e) 1

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(d) $\frac{29}{51}$

(e) 1

Exercise 6 Solution

In a pack, there are 13 Hearts and 39 non-Hearts.

Define
A= event that at least one of them is a Heart
Therefore, $A^c =$ event that neither of them is a Heart.

Define
B= event that first card is not a Heart. So $P(B) = \frac{39}{52} = \frac{3}{4}$.

C= event that second card is not a Heart. So $P(C \mid B) = \frac{38}{51}$

$$\implies P(A^c) = P(B \cap C) = \frac{3}{4} \times \frac{38}{51} = \frac{19}{34}$$

$$\implies P(A) = 1 - P(A^c) = 1 - \frac{19}{34} = \frac{15}{34}$$